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Electromagnetic Tail Radiation in Nonflat Spacetimes

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The Green's function of the electromagnetic wave equation in an arbitrary spacetime is expanded in geodesic coordinates in order to evaluate the tail—the part of the Green's function which does not propagate on the null cone. It is concluded that, to first order in the Riemann tensor, the tail contribution is constant. The electromagnetic power radiated from an accelerated charged particle follows the null cone, i.e., satisfies Huygens' principle, and is the same as if the Riemann tensor were zero. Electromagnetic radiation from compact sources such as neutron stars and black-hole accretion disks may suffer the usual gravitational distortions of the null cone, but none of it arrives slower than "the speed of light."

1. INTRODUCTION

The problem of whether an arbitrary second-order hyperbolic partial differential equation satisfies Huygens' principle remains unsolved. The question is whether the Green's function of a second-order hyperbolic partial differential equation has support only on the null cone itself (i.e., satisfies Huygens' principle) or also on the interior of the null cone. The latter contribution will be called the *tail* of the Green's function. We here restrict our attention to the wave equation in general relativity.

Most investigators have studied the *scalar* wave equation. [For references see Friedlander (1975) and Noonan (1989*a*), designated paper I.] Treatment of the vector and tensor wave equations is rare. The writer used expansions of the wave equation in Robertson's geodesic coordinates to prove that vector and tensor fields satisfy Huygens' principle if and only if the spacetime is flat (Paper I and Noonan, 1989*b*, designated Paper II). A similar conclusion regarding tensor fields was reached by Wünsch (1990).

The purpose of the present paper is to use suitable approximations to evaluate the tail term of the Green's function of the wave equation for the electromagnetic field tensor in general relativity. It is the electromagnetic

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field tensor, not the electromagnetic vector potential, which is chosen for the analysis because the electromagnetic stress-energy tensor (including the Poynting vector) may be obtained directly *without differentiation.* The method used here is the same as used earlier by the writer (Papers I and II), to expand the wave equation in Robertson's geodesic coordinates about the field event, i.e., the spacetime point at which the electromagnetic field is to be evaluated.

The Green's function $G^{\alpha\beta}_{\mu\nu}(P, Q)$ of the wave equation [equation (2) of Paper II]

$$LF^{\mu\nu}=S^{\mu\nu} \tag{1}$$

which is characterized by the solution [equation (8) there]

$$F^{\alpha\beta}(P)=\int \sqrt{g_Q}\, G^{\alpha\beta}_{\mu\nu}(P, Q)S^{\mu\nu}(Q)\, d^4x_Q \tag{2}$$

is defined by the condition

$$L_Q G^{\alpha\beta}_{\mu\nu}(P, Q)=I^{\alpha\beta}_{\mu\nu}(P, Q) \tag{3}$$

where $I^{\alpha\beta}_{\mu\nu}(P, Q)$ is the four-dimensional, second-rank Dirac delta function. [In Paper II, equation (7) has a misprint; $G^{\alpha\beta}_{\mu\nu}$ should be $L_Q G^{\alpha\beta}_{\mu\nu}$.]

2. THE DIFFERENTIAL EQUATION FOR THE TAIL TERM

The Green's function is written

$$G^{\alpha\beta}_{\mu\nu}(P, Q)=E^{\alpha\beta}_{\mu\nu}(P, Q)\, \delta[H(P, Q)]+D^{\alpha\beta}_{\mu\nu}(P, Q) \tag{4}$$

where H is the cone function defined in Paper I. (The choice of the backward null cone, i.e., the rejection of the forward null cone, is dictated by the desire to have the general Green's function reduce to the special-relativity Green's function, which is retarded.) The extra term $D^{\alpha\beta}_{\mu\nu}(P, Q)$ represents the tail of the Green's function. It is omitted in the corresponding equation (9) of Paper II. The substitution of equation (4) into equation (3) gives

$$\begin{aligned} I^{\alpha\beta}_{\mu\nu}(P, Q)=L^{\alpha\beta}_{\mu\nu}(P, Q)\, \delta[H(P, Q)]+N^{\alpha\beta}_{\mu\nu}(P, Q)\, \delta'[H(P, Q)] \\ +E^{\alpha\beta}_{\mu\nu}(P, Q)\, \Delta H(P, Q)\, \delta'' H(P, Q)+T^{\alpha\beta}_{\mu\nu}(P, Q) \end{aligned} \tag{5}$$

where

$$\begin{aligned} L^{\alpha\beta}_{\mu\nu}(P, Q)&=L_Q E^{\alpha\beta}_{\mu\nu}(P, Q) \\ N^{\alpha\beta}_{\mu\nu}(P, Q)&=2[E^{\alpha\beta}_{\mu\nu}(P, Q)]^{;\rho}[H(P, Q)]_{;\rho} \\ &\quad +E^{\alpha\beta}_{\mu\nu}(P, Q)[H(P, Q)]^{;\rho}_{;\rho} \\ \Delta H&=H^{;\rho}H_{;\rho} \\ T^{\alpha\beta}_{\mu\nu}(P, Q)&=L_Q D^{\alpha\beta}_{\mu\nu}(P, Q) \end{aligned} \tag{6}$$

with the covariant derivatives evaluated at Q. The third term in equation (5) will be ignored because the hypersurfaces of constant H are null hypersurfaces. As in Paper I, we here define $K = 1/4\pi r c^2$, where $r^2 = y^k y^k$ in terms of the spatial components y^k of the geodesic coordinates. The special relativity postulate of Paper I requires that $E^{\alpha\beta}_{\mu\nu}$ have the form

$$E^{\alpha\beta}_{\mu\nu}(P, Q) = K[\delta^{\alpha\beta}_{\mu\nu} + f^{\alpha\beta}_{\mu\nu}(P, Q)] \tag{7}$$

where $\delta^{\alpha\beta}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}$ and $f^{\alpha\beta}_{\mu\nu}$ is an analytic, well-behaved function with no poles, and it vanishes in the case of flat spacetime, so that equation (7) reduces to the special-relativity Green's function. Since $f^{\alpha\beta}_{\mu\nu}$ is analytic, it may be expanded in geodesic coordinates:

$$f^{\alpha\beta}_{\mu\nu} = A^{\alpha\beta}_{\mu\nu\rho} y^\rho + \tfrac{1}{2} A^{\alpha\beta}_{\mu\nu\rho\sigma} y^\rho y^\sigma + O^3(y) \tag{8}$$

The expansions of the various functions and equations in geodesic coordinates $y^0 = t$ and y^i proceed exactly as in Paper II to give

$$\begin{aligned}
N^{\alpha\beta}_{\mu\nu} = {}& \frac{t}{r^3}\{-\tfrac{2}{3}\delta^{\alpha\beta}_{\mu\nu} R_{0ij0} y^i y^j + r y^k[-2A^{\alpha\beta}_{\mu\nu k0} + \tfrac{2}{3}\delta^{\beta}_{\nu}(R^{\alpha}{}_{\mu 0k} + R^{\alpha}{}_{k0\mu}) \\
& + \tfrac{2}{3}\delta^{\alpha}_{\mu}(R^{\beta}{}_{\nu 0k} + R^{\beta}{}_{k0\nu})] + r^2(2A^{\alpha\beta}_{\mu\nu 00} - \tfrac{2}{3}\delta^{\beta}_{\nu} R^{\alpha}{}_{00\mu} - \tfrac{2}{3}\delta^{\alpha}_{\mu} R^{\beta}{}_{00\nu})\} \\
& + \frac{y^i y^j}{r^2}[-2A^{\alpha\beta}_{\mu\nu ij} - \tfrac{1}{3}\delta^{\alpha\beta}_{\mu\nu}(R_{0ij0} + 2R_{ij} + R_{kijk}) + \tfrac{2}{3}\delta^{\beta}_{\nu} R^{\alpha}{}_{ij\mu} + \tfrac{2}{3}\delta^{\alpha}_{\mu} R^{\beta}{}_{ij\nu}] \\
& + \frac{y^k}{r}\left[2A^{\alpha\beta}_{\mu\nu 0k} - \frac{2}{r} A^{\alpha\beta}_{\mu\nu k} + \tfrac{2}{3}\delta^{\alpha\beta}_{\mu\nu} R_{0k} - \tfrac{2}{3}\delta^{\beta}_{\nu}(R^{\alpha}{}_{\mu k0} + R^{\alpha}{}_{0k\mu}) \right. \\
& \left. - \tfrac{2}{3}\delta^{\alpha}_{\mu}(R^{\beta}{}_{\nu k0} + R^{\beta}{}_{0k\nu})\right] + \frac{2}{r} A^{\alpha\beta}_{\mu\nu 0} - \tfrac{1}{3}\delta^{\alpha\beta}_{\mu\nu} R_{00}
\end{aligned} \tag{9}$$

$$\begin{aligned}
L^{\alpha\beta}_{\mu\nu} = LE^{\alpha\beta}_{\mu\nu} = {}& -\frac{t^2}{r^5}\delta^{\alpha\beta}_{\mu\nu}(R_{0ij0} y^i y^j + \tfrac{1}{3} r^2 R_{00}) + \frac{t y^k}{r^3}[2A^{\alpha\beta}_{\mu\nu k0} \\
& - \tfrac{2}{3}\delta^{\beta}_{\nu}(R^{\alpha}{}_{\mu 0k} + R^{\alpha}{}_{k0\mu}) - \tfrac{2}{3}\delta^{\alpha}_{\mu}(R^{\beta}{}_{\nu 0k} + R^{\beta}{}_{k0\nu})] \\
& + \frac{y^i y^j}{r^3}[2A^{\alpha\beta}_{\mu\nu ij} + \tfrac{1}{3}\delta^{\alpha\beta}_{\mu\nu}(R_{kijk} + 2R_{ij}) - \tfrac{2}{3}(\delta^{\beta}_{\nu} R^{\alpha}{}_{ij\mu} + \delta^{\alpha}_{\mu} R^{\beta}{}_{ij\nu})] \\
& + \frac{2y^k}{r^3} A^{\alpha\beta}_{\mu\nu k} + \frac{1}{r}[A^{\alpha\beta}_{\mu\nu 00} - A^{\alpha\beta}_{\mu\nu kk} + \tfrac{2}{3}(\delta^{\alpha}_{\mu} R^{\beta}_{\nu} + \delta^{\beta}_{\nu} R^{\alpha}_{\mu}) - 2R^{\alpha}{}_{\mu\nu}{}^{\beta}]
\end{aligned} \tag{10}$$

[For convenience in equations (9) and (10) the speed of light c is taken as unity.]

Equation (16) in Paper II is the same as $r^3 N^{\alpha\beta}_{\mu\nu}$ as given by equation (9) here, with the exception that the $A^{\alpha\beta}_{\mu\nu\rho}$ terms are included rather than omitted. Also, equations (64) of Paper I, which define the Ricci tensor, were used to produce some simplification.

We will retain terms of only first order in the Riemann tensor. It is evident that $f^{\alpha\beta}_{\mu\nu}$ and $D^{\alpha\beta}_{\mu\nu}$ are at least first order in the Riemann tensor or else they would show up in the flat-spacetime Green's function, which they do not. Therefore the product of $f^{\alpha\beta}_{\mu\nu}$ or $D^{\alpha\beta}_{\mu\nu}$ with the Riemann tensor will be omitted. In the expansion of equation (5) in geodesic coordinates, equation (6) becomes

$$\Box D^{\alpha\beta}_{\mu\nu} = T^{\alpha\beta}_{\mu\nu} \tag{11}$$

where $\Box = \eta^{\mu\nu}\, \partial^2/\partial y^\mu\, \partial y^\nu$ and $\eta_{\mu\nu}$ is the special-relativity metric. Thus, in the lowest order the differential equation for the tail $D^{\alpha\beta}_{\mu\nu}$ is the special-relativistic wave equation (11) with the source given by equation (5). Since the left-hand side of equation (5) vanishes everywhere except at $Q=P$, we have

$$T^{\alpha\beta}_{\mu\nu} = -L^{\alpha\beta}_{\mu\nu}\delta(H) - N^{\alpha\beta}_{\mu\nu}\delta'(H) \tag{12}$$

with $L^{\alpha\beta}_{\mu\nu}$ and $N^{\alpha\beta}_{\mu\nu}$ given by equations (9) and (10). Equation (11) is the differential equation to be solved for the tail term $D^{\alpha\beta}_{\mu\nu}$.

3. SOLUTION FOR THE TAIL TERM

The usual procedure in special relativity in solving a wave equation of the form of equation (11) is to use *retarded* potentials. However, here we must use *advanced* potentials for the following reason. It is well known that the Green's function for equation (1) must vanish outside the null cone. Therefore the tail term $D^{\alpha\beta}_{\mu\nu}$ exists only inside the null cone. Consider Figure 1, in which the ordinate is x^0 and the abscissa is a spatial coordinate, say x^1. The *backward* null cone $C(P)$ with vertex P opens downward. It is desired to evaluate $D^{\alpha\beta}_{\mu\nu}$ at a point Q which necessarily lies below (inside) $C(P)$. But according to equation (12) the source of $D^{\alpha\beta}_{\mu\nu}$ lies only on $C(P)$. The only null cone with vertex Q which intersects $C(P)$ is the *forward* null cone $\bar{C}(Q)$, which opens upward. Furthermore, the only contribution to $D^{\alpha\beta}_{\mu\nu}(Q)$ is by $T^{\alpha\beta}_{\mu\nu}(R)$, where R is a point on both $C(P)$ and $\bar{C}(Q)$.

The quantities which are indicated in Figure 1 are constructed as follows (the context is the special-relativistic or flat-spacetime approximation to the

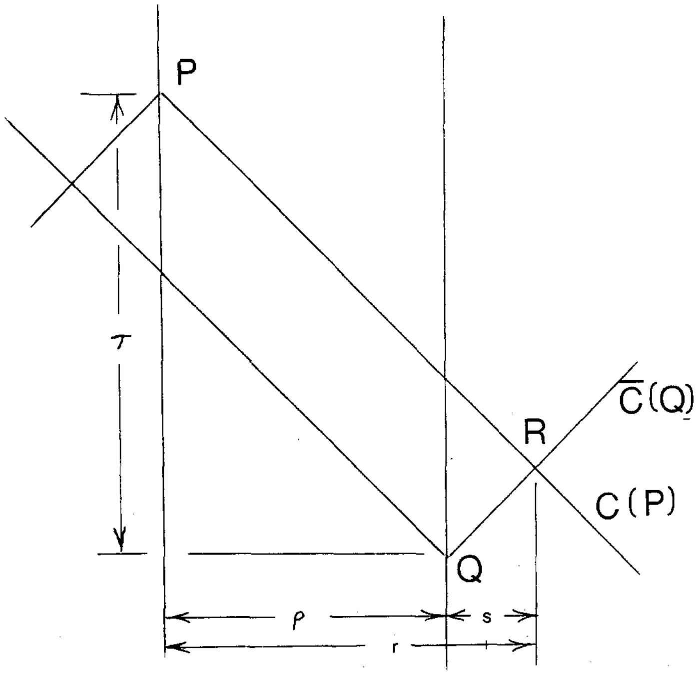

Fig. 1. The relationship between the backward null cone $C(P)$ of the field event P and the forward null cone $\bar{C}(Q)$ of an integration event Q inside $C(P)$.

metric at the field event P):

$$\tau = x_P^0 - x_Q^0$$

ρ = spatial distance between P, Q

r = spatial distance between P, R

s = spatial distance between Q, R

Thus r retains the same meaning as in equations (9) and (10). Since R is on the backward null cone $C(P)$ and on the forward null cone $\bar{C}(Q)$, we have the relation

$$\tau = r + s$$

However we do not have $r = \rho + s$, because Figure 1 suffers the usual defect of trying to portray on a two-dimensional sheet of paper a relation which involves >2 dimensions. The points P, Q, R are not necessarily colinear.

The solution to equation (11) is now

$$D^{\alpha\beta}_{\mu\nu}=\frac{1}{4\pi c^2}\int\frac{1}{s}\,T^{\alpha\beta}_{\mu\nu}\,dV \tag{13}$$

with the integration extending over three-dimensional spatial volume and the arguments of $T^{\alpha\beta}_{\mu\nu}$ are the spatial coordinates $x^i(R)$ and the temporal coordinate $x^0=x^0(Q)+s$.

The next step (omitted here) is the rather laborious task of substituting equations (9), (10), and (12) and then working through the integration of each of the terms in equations (9) and (10). As might be expected, the result is rather cumbersome:

$$\begin{aligned} D^{\alpha\beta}_{\mu\nu}=&\frac{L}{2\rho^3}\left[\frac{3\tau\varepsilon}{\rho^2}y^iy^jB^{\alpha\beta}_{\mu\nu ij}-\tau\varepsilon B^{\alpha\beta}_{\mu\nu kk}-4\tau y^kA^{\alpha\beta}_{\mu\nu k}+\varepsilon y^kC^{\alpha\beta}_{\mu\nu k}\right]\\ &-\frac{3\tau^2-2\rho^2}{\rho^4}y^iy^jB^{\alpha\beta}_{\mu\nu ij}+\frac{\varepsilon}{\rho^2}B^{\alpha\beta}_{\mu\nu kk}\\ &-\frac{\tau}{\rho^2}y^kC^{\alpha\beta}_{\mu\nu k}+\frac{4}{\rho^2}y^kA^{\alpha\beta}_{\mu\nu k}+\Gamma^{\alpha\beta}_{\mu\nu} \end{aligned} \tag{14}$$

where

$$L=\log_e\frac{\tau+\rho}{\tau-\rho}$$

$$\varepsilon=\tau^2-\rho^2$$

$$\begin{aligned} B^{\alpha\beta}_{\mu\nu ij}=&2A^{\alpha\beta}_{\mu\nu ij}+\tfrac{1}{3}\delta^{\alpha\beta}_{\mu\nu}(R_{ij}-2R_{0ij0})-\tfrac{1}{3}[\delta^{\beta}_{\nu}(R^{\alpha}{}_{ij\mu}+R^{\alpha}{}_{ji\mu})\\ &+\delta^{\alpha}_{\mu}(R^{\beta}{}_{ij\nu}+R^{\beta}{}_{ji\nu})] \end{aligned}$$

$$C^{\alpha\beta}_{\mu\nu k}=-2A^{\alpha\beta}_{\mu\nu k0}+\tfrac{2}{3}[\delta^{\beta}_{\nu}(R^{\alpha}{}_{\mu 0k}+R^{\alpha}{}_{k0\mu})+\delta^{\alpha}_{\mu}(R^{\beta}{}_{\nu 0k}+R^{\beta}{}_{k0\nu})]$$

$$\Gamma^{\alpha\beta}_{\mu\nu}=-A^{\alpha\beta}_{\mu\nu\rho\sigma}\eta^{\rho\sigma}-\tfrac{1}{6}(\delta^{\alpha}_{\mu}R^{\beta}_{\nu}+\delta^{\beta}_{\nu}R^{\alpha}_{\mu})+R^{\alpha}{}_{\mu\nu}{}^{\beta}+\tfrac{1}{12}\delta^{\alpha\beta}_{\mu\nu}(4R_{00}-R)$$

$$R=\eta^{\rho\sigma}R_{\rho\sigma}$$

Equation (14) is the solution to equation (11). It must be kept in mind that, in the absence of boundary conditions, an arbitrary constant can be added to the solution of equation (11). Thus, the constant term $\Gamma^{\alpha\beta}_{\mu\nu}$ in equation (14) may turn out to be an illusion. Equation (14) results from a strict application of the special-relativity Green's function to equation (11).

4. TRANSFORMATION CONDITIONS ON THE TAIL TERM

The requirement that the tail term $D^{\alpha\beta}_{\mu\nu}$ must transform as a tensor due to the tensor character of $G^{\alpha\beta}_{\mu\nu}$, at least in the special-relativistic application which is valid for the geodesic coordinates being used, imposes conditions on the coefficients $A^{\alpha\beta}_{\mu\nu\rho}$ and $A^{\alpha\beta}_{\mu\nu\rho\sigma}$ which were introduced in equation (8). The task of performing an infinitesimal coordinate transformation of equation (14) is a chore whose tediousness ranks on the same level as the derivation of equation (14). The results are

$$A^{\alpha\beta}_{\mu\nu\rho}=0 \tag{15}$$

$$\begin{aligned} A^{\alpha\beta}_{\mu\nu\rho\sigma}=\delta^{\alpha\beta}_{\mu\nu}[\tfrac{1}{3}R_{0\rho\sigma0}-\tfrac{1}{6}R_{\rho\sigma}+\tfrac{1}{6}(\eta_{0\sigma}R_{0\rho}+\eta_{0\rho}R_{0\sigma})] \\ +\tfrac{1}{2}J^{\alpha\beta}_{\mu\nu\rho\sigma}+\tfrac{1}{2}(\eta_{0\sigma}K^{\alpha\beta}_{\mu\nu\rho0}+\eta_{0\rho}K^{\alpha\beta}_{\mu\nu\sigma0})+M^{\alpha\beta}_{\mu\nu}\eta_{\rho\sigma} \end{aligned} \tag{16}$$

where

$$J^{\alpha\beta}_{\mu\nu\rho\sigma}=\tfrac{1}{3}[\delta^{\beta}_{\nu}(R^{\alpha}{}_{\rho\sigma\mu}+R^{\alpha}{}_{\sigma\rho\mu})+\delta^{\alpha}_{\mu}(R^{\beta}{}_{\rho\sigma\nu}+R^{\beta}{}_{\sigma\rho\nu})]$$

$$K^{\alpha\beta}_{\mu\nu\rho\sigma}=\delta^{\beta}_{\nu}R^{\alpha}{}_{\mu\sigma\rho}+\delta^{\alpha}_{\mu}R^{\beta}{}_{\nu\sigma\rho}$$

$$M^{\alpha\beta}_{\mu\nu}=\text{set of undetermined constants}$$

The substitution of equations (15) and (16) into equation (14) gives

$$D^{\alpha\beta}_{\mu\nu}=\Gamma^{\alpha\beta}_{\mu\nu}+2M^{\alpha\beta}_{\mu\nu}$$

Thus the tail term $D^{\alpha\beta}_{\mu\nu}$ is a constant on the interior of the null cone and zero on the exterior of the null cone, and it is discontinuous at the null cone.

5. THE CONIC CONTRIBUTION FOR A CHARGED PARTICLE

According to equations (2), (4), (7), (8), and (15), we now have

$$F^{\alpha\beta}(P)=\int\sqrt{g_Q}\,S^{\mu\nu}(Q)\,d^4x_Q\left[\frac{1}{4\pi rc^2}\,\delta(H)(\delta^{\alpha\beta}_{\mu\nu}+\tfrac{1}{2}A^{\alpha\beta}_{\mu\nu\rho\sigma}y^\rho y^\sigma)+D^{\alpha\beta}_{\mu\nu}\right] \tag{17}$$

In geodesic coordinates we have

$$\sqrt{g}=1+\tfrac{1}{6}R_{\rho\sigma}y^\rho y^\sigma+O^3(y)$$

Thus equation (17) becomes

$$F^{\alpha\beta}(P)=\int S^{\mu\nu}(y)\,d^4y\left[\frac{1}{4\pi rc^2}\,\delta(H)U^{\alpha\beta}_{\mu\nu}+D^{\alpha\beta}_{\mu\nu}\right] \tag{18}$$

where

$$U^{\alpha\beta}_{\mu\nu}=\delta^{\alpha\beta}_{\mu\nu}+\tfrac{1}{2}V^{\alpha\beta}_{\mu\nu\rho\sigma}y^\rho y^\sigma \tag{19}$$

$$V^{\alpha\beta}_{\mu\nu\rho\sigma}=A^{\alpha\beta}_{\mu\nu\rho\sigma}+\tfrac{1}{3}\delta^{\alpha\beta}_{\mu\nu}R_{\rho\sigma} \tag{20}$$

The integrand in equation (18) contains two terms. The first term represents an integration over the null cone. It represents the contribution to the field tensor which satisfies Huygens' principle, i.e., propagates on null geodesics. The second term represents an integration over the interior of the null cone. We shall treat these two effects in turn, first in this section the conic contribution and then in the next section the interior contribution.

For the remainder of this section we consider equation (18) in the form

$$F^{\alpha\beta}=\frac{1}{4\pi c^2}\int W^{\alpha\beta}\frac{1}{r}\,\delta(H)\,d^4y \tag{21}$$

where

$$W^{\alpha\beta}=S^{\mu\nu}U^{\alpha\beta}_{\mu\nu} \tag{22}$$

Equation (21) may be immediately integrated over dy^0 to give

$$F^{\alpha\beta}=\frac{1}{4\pi c^2}\int W^{\alpha\beta}\frac{1}{r}\,d^3y \tag{23}$$

In our application to the case of a charged particle, we follow the custom of Robertson and of the writer by regarding the particle to be a world-tube which is shrunk to a world-line. It is emphasized that we are here concerned only with the particle's contribution to the right-hand side of equation (1). Other details, such as the equation of motion of the particle due to external electromagnetic fields and its own radiation reaction, and the particle's internal structure and rigidity, are not touched. Figure 2 shows the geometry. Before the particle is shrunk to a world-line, its "center" has world-line L (to which the world-tube will be shrunk), and another part of the particle has world-line L'. The field event is P, and the backward null cone from P is C. The world-line L intersects C at O. The spatial three-dimensional space contemporary with O is V. The world-line L' intersects C at Q and intersects V at A.

Although the integration in equation (23) is over Q, it is easier to perform the integration in the space V. Therefore it is necessary to transform the various quantities from the event Q to the event A. The following spatial position vectors will be used:

R^i = position of P relative to O

z^i = position of Q relative to O

r^i = position of P relative to Q

w^i = position of A relative to O

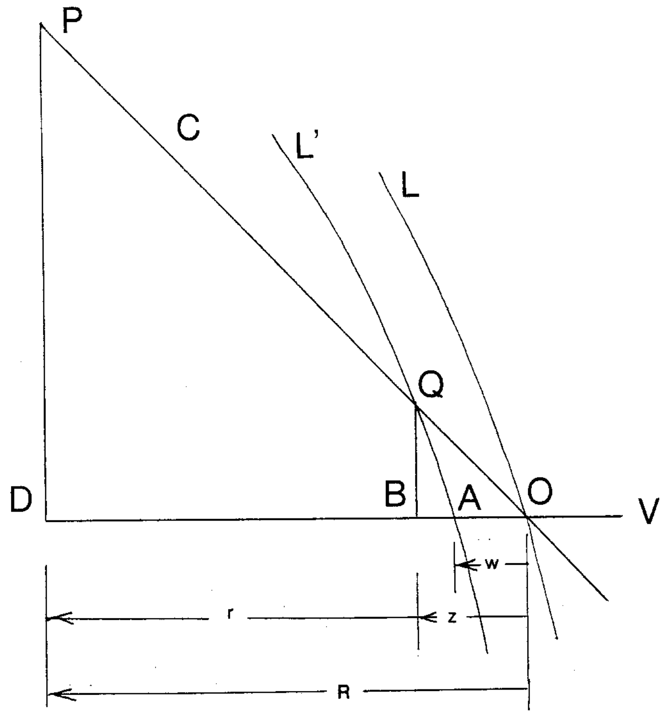

Fig. 2. The intersection of the world-lines of a charged particle with the backward null cone of the field event. The field event is P and its backward null cone is C. Two of the particle's world-lines L and L' intersect C at O and Q, respectively.

Define also

$$T = \frac{R - r}{c} \tag{24}$$

where $R^2 = R^k R^k$ and $r^2 = r^k r^k$. That is, T is the time from O to Q. The motion $x^i(t)$ of L' is expanded in a Taylor series to give

$$z^i = w^i + v^i T + \tfrac{1}{2} a^i T^2 + \cdots \tag{25}$$

where v^i is the velocity of L' at A and a^i is the acceleration of L' at A. We choose the Lorentz frame in which the velocity of L at O is zero. Then the velocity of L' at A is also zero, and equation (25) becomes

$$z^i = w^i + \tfrac{1}{2} a^i T^2 + \cdots \tag{26}$$

From the relation

$$r^i = R^i - z^i \tag{27}$$

we have

$$r = R\left[1 - \frac{R^k z^k}{R^2} + O^2(z)\right] \tag{28}$$

The substitution of equation (28) into equation (24) gives

$$T = \frac{R^k z^k}{Rc} + O^2(z) \tag{29}$$

The substitution of equation (29) into equation (26) gives

$$z^i = w^i + \frac{a^i (R^k z^k)^2}{2R^2 c^2} + O^3(z) \tag{30}$$

Equation (30) provides the relation between w^i and z^i. We next replace the z^i in equations (28) and (29) with w^i. Multiply equation (30) by R^i:

$$R^k z^k = R^k w^k + O^2(z) \tag{31}$$

Substitute equation (31) into equations (28)–(30), respectively. We have

$$r = R\left[1 - \frac{R^k w^k}{R^2} + O^2(w)\right] \tag{32}$$

$$T = \frac{R^k w^k}{Rc} + O^2(w) \tag{33}$$

$$z^i = w^i + \frac{a^i (R^k w^k)^2}{2R^2 c^2} + O^3(w) \tag{34}$$

From equation (32) we have

$$\frac{1}{r} = \frac{1}{R}\left[1 + \frac{R^k w^k}{R^2} + O^2(w)\right] \tag{35}$$

There is the task of transforming the volume element in equation (23). We need to map the volume element $dV' = d^3 y$ at Q into the volume element dV at A. This requires the use of the Jacobian of equation (34):

$$\frac{\partial z^i}{\partial w^k} = \delta^i_k + \frac{a^i R^k R^m w^m}{R^2 c^2} + O^2(w)$$

which has determinant

$$\frac{dV'}{dV}=\left|\frac{\partial z}{\partial w}\right|=1+\frac{a^k R^k R^m w^m}{R^2 c^2}+O^2(w) \tag{36}$$

There remains the problem that equation (23) requires that $W^{\alpha\beta}$ be evaluated at Q rather than at A. Therefore we must replace $W^{\alpha\beta}$ in equation (23) by

$$W_Q^{\alpha\beta}=W^{\alpha\beta}+T\frac{dW^{\alpha\beta}}{dt}=W^{\alpha\beta}+\frac{R^k w^k}{Rc}\frac{dW^{\alpha\beta}}{dt}+O^2(w) \tag{37}$$

where the $W^{\alpha\beta}$ and $dW^{\alpha\beta}/dt$ on the right-hand side of equation (37) are evaluated at A. From equation (22) we have

$$\frac{dW^{\alpha\beta}}{dt}=\frac{dS^{\mu\nu}}{dt}U_{\mu\nu}^{\alpha\beta}+S^{\mu\nu}\frac{dU_{\mu\nu}^{\alpha\beta}}{dt} \tag{38}$$

Since the tensor $S^{\mu\nu}$ is carried by parallel transport, the quantity $dS^{\mu\nu}/dt$ is the Thomas precession and may be written

$$\frac{dS^{\mu\nu}}{dt}=X_{\rho\sigma}^{\mu\nu}S^{\rho\sigma} \tag{39}$$

The substitution of equations (22), (38), and (39) into equation (37) gives

$$W_Q^{\alpha\beta}=S^{\mu\nu}\left[U_{\mu\nu}^{\alpha\beta}+\frac{R^k w^k}{Rc}\left(X_{\mu\nu}^{\rho\sigma}U_{\rho\sigma}^{\alpha\beta}+\frac{dU_{\mu\nu}^{\alpha\beta}}{dt}\right)\right]+O^2(w) \tag{40}$$

The next step is to find suitable expressions for $U_{\mu\nu}^{\alpha\beta}$ and $dU_{\mu\nu}^{\alpha\beta}/dt$ to substitute into equation (40). From equation (19) we have

$$U_{\mu\nu}^{\alpha\beta}=\delta_{\mu\nu}^{\alpha\beta}+\tfrac{1}{2}[V_{\mu\nu00}^{\alpha\beta}(y^0)^2+2V_{\mu\nu k0}^{\alpha\beta}y^0y^k+V_{\mu\nu km}^{\alpha\beta}y^ky^m] \tag{41}$$

We shall take the derivative of equation (41) with respect to $t=y^0$. There arise the derivatives $v^i=dy^i/dt$. However, we have chosen a Lorentz frame in which $v^i=0$. Therefore such terms will be omitted. The result is

$$\frac{dU_{\mu\nu}^{\alpha\beta}}{dt}=V_{\mu\nu00}^{\alpha\beta}y^0+V_{\mu\nu k0}^{\alpha\beta}y^k \tag{42}$$

Since y^μ is measured from P to Q, but r^i is measured from Q to P, it is necessary to substitute

$$\begin{aligned} y^0&=-\frac{r}{c}=-\frac{R}{c}\left(1-\frac{R^k w^k}{R^2}\right)\\ y^i&=-r^i=-(R^i-w^i)\end{aligned} \tag{43}$$

Since $dU^{\alpha\beta}_{\mu\nu}/dt$ enters equations (40) in a term which is already of order w, equations (43) in the form $y^0 = -R/c$ and $y^i = -R^i$ may be used:

$$\frac{dU^{\alpha\beta}_{\mu\nu}}{dt} = -c^{-1} V^{\alpha\beta}_{\mu\nu 00} R - V^{\alpha\beta}_{\mu\nu k0} R^k \tag{44}$$

However, the whole of equations (43) must be used in $U^{\alpha\beta}_{\mu\nu}$. The substitution of equations (43) into equation (41) produces three kinds of terms:

$$U^{\alpha\beta}_{\mu\nu} = \delta^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta}_{\mu\nu} - w^k Z^{\alpha\beta}_{\mu\nu k} \tag{45}$$

where

$$\Lambda^{\alpha\beta}_{\mu\nu} = \tfrac{1}{2}[c^{-2} V^{\alpha\beta}_{\mu\nu 00} R^2 + 2c^{-1} V^{\alpha\beta}_{\mu\nu k0} R R^k + V^{\alpha\beta}_{\mu\nu km} R^k R^m] \tag{46}$$

$$Z^{\alpha\beta}_{\mu\nu k} = c^{-2} V^{\alpha\beta}_{\mu\nu 00} R^k + c^{-1} V^{\alpha\beta}_{\mu\nu k0} R + c^{-1} V^{\alpha\beta}_{\mu\nu m0} R^{-1} R^m R^k + V^{\alpha\beta}_{\mu\nu km} R^m \tag{47}$$

Substitute equation (45) into equation (40),

$$W^{\alpha\beta}_Q = S^{\mu\nu}\left\{\delta^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta}_{\mu\nu} + w^k\left[-Z^{\alpha\beta}_{\mu\nu k} + \frac{R^k}{Rc}\left(X^{\alpha\beta}_{\mu\nu} + X^{\rho\sigma}_{\mu\nu}\Lambda^{\alpha\beta}_{\rho\sigma} + \frac{dU^{\alpha\beta}_{\mu\nu}}{dt}\right)\right]\right\} + O^2(w) \tag{48}$$

Substitute equations (35), (36), and (48) into equation (23),

$$4\pi c^2 R F^{\alpha\beta} = \int dV\, S^{\mu\nu}\left\{\delta^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta}_{\mu\nu} - w^k Z^{\alpha\beta}_{\mu\nu k} + \frac{R^k w^k}{R^2}\right.$$
$$\left.\times\left[\frac{R}{c}\left(X^{\alpha\beta}_{\mu\nu} + X^{\rho\sigma}_{\mu\nu}\Lambda^{\alpha\beta}_{\rho\sigma} + \frac{dU^{\alpha\beta}_{\mu\nu}}{dt}\right) + \left(1 + \frac{a^m R^m}{c^2}\right)(\delta^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta}_{\mu\nu})\right]\right\} \tag{49}$$

In the corpuscular limit as the particle's world-tube shrinks to a world-line, equation (49) becomes

$$4\pi c^2 R F^{\alpha\beta} = \sigma^{\alpha\beta} + \sigma^{\mu\nu}\Lambda^{\alpha\beta}_{\mu\nu} + \tau^{\mu\nu k}\left\{-Z^{\alpha\beta}_{\mu\nu k} + \frac{R^k}{R^2}\right.$$
$$\left.\times\left[\frac{R}{c}\left(X^{\alpha\beta}_{\mu\nu} + X^{\rho\sigma}_{\mu\nu}\Lambda^{\alpha\beta}_{\rho\sigma} + \frac{dU^{\alpha\beta}_{\mu\nu}}{dt}\right) + \left(1 + \frac{a^m R^m}{c^2}\right)(\delta^{\alpha\beta}_{\mu\nu} + \Lambda^{\alpha\beta}_{\mu\nu})\right]\right\} \tag{50}$$

where

$$\begin{aligned} \sigma^{\mu\nu} &= \int S^{\mu\nu}\, dV \\ \tau^{\mu\nu k} &= \int S^{\mu\nu} w^k\, dV \end{aligned} \tag{51}$$

The source tensor $S^{\mu\nu}$ which appears in equations (1) and (2) is

$$S^{\mu\nu} = \varepsilon_0^{-1}(J^{\mu;\nu} - J^{\nu;\mu}) \tag{52}$$

For the special-relativity context of the present treatment this is

$$S^{0i} = -S^{i0} = -\frac{1}{\varepsilon_0}\left[c^2\frac{\partial\rho}{\partial x^i} + \frac{\partial}{\partial t}(\rho v^i)\right] \tag{53}$$

$$S^{ij} = \frac{c^2}{\varepsilon_0}\left[\frac{\partial}{\partial x^i}(\rho v^j) - \frac{\partial}{\partial x^j}(\rho v^i)\right] \tag{54}$$

where ρ is the charge density and v^i is its velocity. Conservation of charge implies

$$\frac{\partial}{\partial t}(\rho v^i) = \rho a^i - \frac{\partial}{\partial x^k}(\rho v^i v^k)$$

where $a^i = dv^i/dt$ is the acceleration. Thus equation (53) becomes

$$S^{0i} = -\frac{1}{\varepsilon_0}\left\{\frac{\partial}{\partial x^k}[\rho(c^2\delta^{ik} - v^i v^k)] + \rho a^i\right\} \tag{55}$$

When equations (54) and (55) are substituted into equations (51), four kinds of integrals arise:

(i) $\int \rho a^i \, dV$.
(ii) $\int \rho a^i w^j \, dV$.
(iii) $\int (\partial Q/\partial x^i) \, dV$.
(iv) $\int (\partial Q/\partial x^i) w^j \, dV$.

where Q is a quantity which includes the charge density. Terms of higher order in w^i are neglected because of the character of the expansions. Case (i) becomes in the corpuscular limit qa^i, where q is the total charge. Case (ii) is a dipole term which will be neglected. Case (iii) is transformed by Gauss' theorem into a surface integral outside the particle's world-tube, where its density is zero. Case (iv) is transformed by Gauss' theorem:

$$\int\left[\frac{\partial}{\partial x^i}(Qw^j) - Q\frac{\partial w^j}{\partial x^i}\right] dV = -\delta^j_i \int Q \, dV$$

Thus, equations (51) become

$$\sigma^{0i} = -\frac{1}{\varepsilon_0} q a^i \tag{56a}$$

$$\sigma^{ij} = 0 \tag{56b}$$

$$\tau^{0ik} = \frac{q}{\varepsilon_0} (c^2 \delta^{ik} - v^i v^k) \tag{56c}$$

$$\tau^{ijk} = 0 \tag{56d}$$

However, we are using the Lorentz frame in which $v^i = 0$, giving

$$\tau^{0ik} = \frac{qc^2}{\varepsilon_0} \delta^{ik} \tag{56e}$$

The substitution of equations (56) into equation (50) gives

$$\begin{aligned}
\frac{4\pi c^2 \varepsilon_0 R F^{0i}}{q} &= -a^i - a^k \bar{\Lambda}^{0i}_{0k} - c^2 \bar{Z}^{0i}_{0kk} \\
&\quad + \frac{cR^k}{R} \left(\bar{X}^{0i}_{0k} + \bar{X}^{\rho\sigma}_{0k} \Lambda^{0i}_{\rho\sigma} + \frac{d\bar{U}^{0i}_{0k}}{dt} \right) \\
&\quad + (c^2 + a^m R^m) \left(\frac{R^i}{R^2} + \frac{R^k}{R^2} \bar{\Lambda}^{0i}_{0k} \right) \\
\frac{4\pi c^2 \varepsilon_0 R F^{hi}}{q} &= -a^k \bar{\Lambda}^{hi}_{0k} - c^2 \bar{Z}^{hi}_{0kk} \\
&\quad + \frac{cR^k}{R} \left(\bar{X}^{hi}_{0k} + \bar{X}^{\rho\sigma}_{0k} \Lambda^{hi}_{\rho\sigma} + \frac{d\bar{U}^{hi}_{0k}}{dt} \right) \\
&\quad + (c^2 + a^m R^m) \bar{\Lambda}^{hi}_{0k} \frac{R^k}{R^2}
\end{aligned} \tag{57}$$

where

$$\bar{\Lambda}^{\alpha\beta}_{\mu\nu} = \Lambda^{\alpha\beta}_{\mu\nu} - \Lambda^{\alpha\beta}_{\nu\mu}$$

$$\bar{Z}^{\alpha\beta}_{\mu\nu k} = Z^{\alpha\beta}_{\mu\nu k} - Z^{\alpha\beta}_{\nu\mu k}$$

$$\bar{X}^{\alpha\beta}_{\mu\nu} = X^{\alpha\beta}_{\mu\nu} - X^{\alpha\beta}_{\nu\mu}$$

$$\bar{U}^{\alpha\beta}_{\mu\nu} = U^{\alpha\beta}_{\mu\nu} - U^{\alpha\beta}_{\nu\mu}$$

Evaluation of the Thomas precession by the use of the expressions in Robertson and Noonan (1968) gives for the quantity defined in equation (39) in the rest frame

$$\bar{X}^{0i}_{0k} = 0$$
$$\bar{X}^{0i}_{km} = c^{-2}\delta^{i}_{m}a^{k}$$
$$\bar{X}^{hi}_{0k} = \delta^{i}_{k}a^{h} - \delta^{h}_{k}a^{i}$$

Thus equations (57) become

$$\frac{4\pi c^2\varepsilon_0 RF^{\alpha\beta}}{q} = \frac{4\pi c^2\varepsilon_0 RF^{\alpha\beta}_C}{q} - a^k\bar{\Lambda}^{\alpha\beta}_{0k} - c^2\bar{Z}^{\alpha\beta}_{0kk} + \frac{cR^k}{R}\left(a^m\bar{\Lambda}^{\alpha\beta}_{mk} + \frac{d\bar{U}^{\alpha\beta}_{0k}}{dt}\right) + (c^2 + a^m R^m)\frac{R^k}{R^2}\bar{\Lambda}^{\alpha\beta}_{0k} \tag{58}$$

where

$$F^{0i}_C = \frac{q}{4\pi c^2\varepsilon_0 R^3}[c^2R^i + R^k(R^i a^k - R^k a^i)] \tag{59}$$

$$F^{hi}_C = \frac{q}{4\pi c\varepsilon_0 R^2}(R^i a^h - R^h a^i) \tag{60}$$

Equation (58) forms the principal result of this section. For the left-hand side we have in the special-relativity limit

$$\begin{aligned} F^{0i} &= E_i \\ F^{hi} &= c^2 e(hij)B_j \end{aligned} \tag{61}$$

(Robertson and Noonan, 1968), where E_i is the electric field intensity and B_i is the magnetic induction. The right-hand side of equation (58) contains two parts. The classical (non-Riemannian) result is represented by $F^{\alpha\beta}_C$, which is given by equations (59) and (60). The remaining terms in equation (58) represent the effect of a nonzero Riemann tensor. The Riemann effect is included by using equation (16) to find $A^{\alpha\beta}_{\mu\nu\rho\sigma}$, which in turn is used in equation (20) to obtain $V^{\alpha\beta}_{\mu\nu\rho\sigma}$. The latter is used in equations (46) and (47) to find $\Lambda^{\alpha\beta}_{\mu\nu}$ and $Z^{\alpha\beta}_{\mu\nu k}$, which (in their antisymmetric forms $\bar{\Lambda}^{\alpha\beta}_{\mu\nu}$ and $\bar{Z}^{\alpha\beta}_{\mu\nu k}$) appear in equation (58).

The classical terms deserve special comment. From equations (59)–(61) we have

$$E_i = \frac{q}{4\pi\varepsilon_0 R^3}\left[R^i + \frac{R^k}{c^2}(R^i a^k - R^k a^i)\right] \tag{62}$$

$$e(hij)B_j = \frac{\mu_0 q}{4\pi c R^2}(R^i a^h - R^h a^i) \tag{63}$$

These are the expressions for the electric field and the magnetic field of an accelerated charged particle in the rest frame as given by Panofsky and Phillips (1955, pp. 299, 300). The first term in equation (62) is the Coulomb force. Equation (63) and the second term in equation (62) are the radiation terms which represent the radiation from an accelerated charge. Two features of the classical equations (62) and (63) may be noted. First, it is seen in the derivation that the magnetic field of equation (63) arises from the Thomas precession of the charged particle. Without the Thomas precession the magnetic field would vanish. Second, to the writer's knowledge, the present treatment is the *only* derivation of the electric field and the magnetic field of an accelerated charge. The usual route is to first derive the electromagnetic vector potential of an accelerated charge and then differentiate it.

Equation (58) appears to suffer a divergence problem. When one considers the dependence of the various quantities on R, the distance of the field point from the charged particle, remembering that $Z^{\alpha\beta}_{\mu\nu k}$ and $dU^{\alpha\beta}_{\mu\nu}/dt$ vary with R and $\Lambda^{\alpha\beta}_{\mu\nu}$ varies with R^2, one finds terms which vary as R^{-1}, R^0, R^1, and R^2. It is the latter category of term which appears to create a divergence, i.e., E_i and B_i would increase with R. This situation is a result of the expansion in geodesic coordinates where each successive term has a higher power of the coordinates. This problem will be avoided by evaluating various quantities at small values of R.

6. THE INTERIOR CONTRIBUTION FOR A CHARGED PARTICLE

The preceding section treated the first term in the integrand of equation (18). The present section treats the second term. We write

$$F^{\alpha\beta}=\int S^{\mu\nu}D^{\alpha\beta}_{\mu\nu}\,d^4y \tag{64}$$

Substitute equation (52) in its special-relativity form and define

$$\bar{D}^{\alpha\beta}_{\mu\nu}=D^{\alpha\beta}_{\mu\nu}-D^{\alpha\beta}_{\nu\mu}$$

Equation (64) becomes

$$\begin{aligned}F^{\alpha\beta}&=\frac{1}{\varepsilon_0}\int J^{\mu,\nu}\bar{D}^{\alpha\beta}_{\mu\nu}\,d^4y\\&=\frac{1}{\varepsilon_0}\int[(J^{\mu}\bar{D}^{\alpha\beta}_{\mu\nu})^{,\nu}-J^{\mu}\bar{D}^{\alpha\beta,\nu}_{\mu\nu}]\,d^4y\end{aligned} \tag{65}$$

Equation (65) applies to an arbitrary charge distribution. The first term in the integrand may be transformed by means of Gauss' theorem into a surface integral over a three-dimensional hypersurface which encloses the backward null cone. Thus the only contribution to the first term is from the charge distribution inside the null cone in the remote past, presumably beyond the region of validity of the approximations used here. The second term in the integrand of equation (65) is nonzero only on the null cone, because $D_{\mu\nu}^{\alpha\beta}$ is constant inside and outside the null cone and is discontinuous at the null cone. Thus the second term contributes to the conic field, i.e., the field which propagates on the null cone and satisfies Huygens' principle. It is concluded that the only non-Huygensian contribution to the electromagnetic field is from the charge distribution in the remote past, too early for the expansion in geodesic coordinates to be valid.

For the case of a charged particle, we may evaluate the first term in equation (65) as follows. The Gaussian surface is chosen to be a cylinder surrounding the particle's world-tube with one end of the cylinder above (future from) the null cone and the other end far below (in the remote past from) the null cone. The sides of the cylinder produce no contribution because $J^\mu = 0$ outside the particle's world-tube. The top end of the cylinder produces no contribution because $D_{\mu\nu}^{\alpha\beta} = 0$ outside the null cone. The only contribution to the first term in equation (65) comes from an integral of the form

$$-\frac{1}{\varepsilon_0}\bar{D}_{\mu 0}^{\alpha\beta}\int J^\mu\, dV = -\frac{1}{\varepsilon_0}\bar{D}_{k0}^{\alpha\beta} q v^k \tag{66}$$

where v^k is the velocity of the charge at some remote time in the past. This term will drop out when we later go to the limit $R \to 0$. In any case equation (66) depends on the particle's velocity rather than its acceleration and would presumably contribute to the electromagnetic field of a nonaccelerated charged particle in a gravitational field.

For the second term in equation (65) we have the integral

$$F^{\alpha\beta} = -\frac{1}{\varepsilon_0}\int J^\mu \bar{D}_{\mu\nu}^{\alpha\beta,\nu}\, d^4y$$

where $\bar{D}_{\mu\nu}^{\alpha\beta}$ is a function of the cone function $H = t + r$ with a discontinuity at $H = 0$. Integration past the discontinuity gives

$$F^{\alpha\beta} = \frac{1}{\varepsilon_0}\bar{D}_{\mu 0}^{\alpha\beta}\int J^\mu\, d^3y - \frac{c}{\varepsilon_0}\bar{D}_{\mu k}^{\alpha\beta}\int J^\mu \frac{y^k}{r}\, d^3y$$

where $\bar{D}_{\mu\nu}^{\alpha\beta}$ is constant and the integration is over the spatial volume d^3y for the integrands evaluated on the null cone. If one follows an analysis similar to what was done in the previous section, one would transform the volume element by means of equation (34), replace y^k with $-R^k+w^k$, use equation (35) for $1/r$, and replace J^μ by $J^\mu+T\,dJ^\mu/dt$. However, when dipole terms of the form $\int \rho w^i\,dV$ are neglected (and remembering to use the particle's rest frame) the result is

$$F^{\alpha\beta}=\frac{qcR^k}{\varepsilon_0 R}\bar{D}^{\alpha\beta}_{0k} \tag{67}$$

Equation (67) is the part of the electromagnetic field which is due to the tail term of the Green's function and propagated on the null cone.

The results of this section may be summarized as follows:

$$\frac{\varepsilon_0 F^{\alpha\beta}}{q}=\bar{D}^{\alpha\beta}_{0k}\left(\frac{cR^k}{R}+v^k\right) \tag{68}$$

We conclude this section by gathering together the results of this and the previous section. The electromagnetic field $F^{\alpha\beta}$ is given by

$$F^{\alpha\beta}=F^{\alpha\beta}_C+F^{\alpha\beta}_R \tag{69}$$

where $F^{\alpha\beta}_C$ is the classical term given by equations (59) and (60) and $F^{\alpha\beta}_R$ contains the Riemannian terms. The latter includes equation (68) and the nonclassical terms in equation (58).

7. THE POYNTING VECTOR

We shall evaluate the Poynting vector P_i in the neighborhood of the charged particle, i.e., for small values of the distance R between the field point and the particle. The flat spacetime of special relativity is adequate. The Poynting vector is given by (Robertson and Noonan, 1968)

$$P_i=c^2\varepsilon_0\eta_{\rho\sigma}F^{0\rho}F^{\sigma i}=-\varepsilon_0 F^{0k}F^{ki} \tag{70}$$

[For the more familiar relation, substitute equation (61) and $c^2\varepsilon_0=\mu_0^{-1}$ to obtain $P_i=\mu_0^{-1}e(ijk)E_jB_k$.] Substitute equation (69) into equation (70) and neglect terms which are quadratic in the Riemann tensor,

$$P_i={}_cP_i+{}_aP_i+{}_bP_i$$

where

$${}_cP_i=-\varepsilon_0 F^{0k}_C F^{ki} \tag{71}$$

$${}_aP_i=-\varepsilon_0 F^{0k}_C F^{ki}_R \tag{72}$$

$${}_bP_i=-\varepsilon_0 F^{0k}_R F^{ki}_C \tag{73}$$

Equation (71) is the classical Poynting vector of an accelerated charge in its rest frame. The substitution of equations (59) and (60) gives

$$ {}_cP_i=\frac{q^2}{16\pi^2c^3\varepsilon_0R^5}\{c^2[R^2a^i-R^iR^ka^k]+R^i[R^2a^2-(R^ka^k)^2]\} \tag{74}$$

The first bracket in equation (74) is a spatial vector perpendicular to the radius R^i. It represents an energy flux associated with the electrostatic Coulomb field. The second bracket in equation (74) is parallel to the radius R^i. The radial component of equation (74),

$$\frac{R^k{}_cP_k}{R}=\frac{q^2}{16\pi^2c^3\varepsilon_0R^4}[R^2a^2-(R^ka^k)^2]$$

is integrated over a sphere of radius R to give the classical radiated power

$$\frac{q^2a^2}{6\pi c^3\varepsilon_0}$$

The use of equation (59) and the antisymmetry of F_R^{ki} give for the radial component of equation (72)

$$ {}_aP_R=\frac{R^k}{R}\,{}_aP_k=\frac{qa^kR^iF_R^{ki}}{4\pi c^2R^2} \tag{75}$$

Equation (60) can be used to write the radial component of equation (73) in the form

$$ {}_bP_R=\frac{R^k}{R}\,{}_bP_k=\frac{q}{4\pi cR^3}(R^kR^ma^m-R^2a^k)F_R^{0k} \tag{76}$$

According to equations (44), (46), (47), (58), and (68), $F_R^{\alpha\beta}$ contains terms which vary as R^0 and R^1. Thus equations (75) and (76) contain terms which vary as R^{-1} and R^0. Multiplication of equations (75) and (76) by the area element $R^2\,d\Omega$, where $d\Omega$ is the solid angle, produces terms which vary as R^1 and R^2. In the limit $R\to 0$ these terms go to zero. It is concluded that the power radiated from a charged particle which is accelerated in a spacetime region of nonzero Riemann tensor is, to first order in the Riemann tensor, independent of the Riemann tensor.

This is not the same as saying that the Riemann tensor produces *no* effect on the Poynting vector. The Riemann tensor does alter the energy flux near the accelerated charged particle. But the altered flux has the same effect as the classical Coulomb flux—it has no net radial component.

8. CONCLUSION

The Green's function of the second-rank electromagnetic wave equation was expanded in geodesic coordinates subject to two conditions: (a) it reduces for flat spacetime to the special-relativity Green's function; (b) only terms of first order in the Riemann tensor are retained; quadratic and higher terms are neglected. The tail of the Green's function in nonflat spacetime is nonzero but is constant. Thus the electromagnetic fields at a given event are affected by the tail in two ways. (a) There is the part due to the discontinuity of the tail at the null cone; it propagates along the null cone so as to become a part of the Green's function which has support on the null cone, i.e., which satisfies Huygens' principle. (b) There is a contribution from the interior of the null cone in the remote past, beyond the range of the validity of the expansions which were used.

For the case of an accelerated charged particle the power emitted is, to first order in the Riemann tensor, the same as for zero Riemann tensor, i.e., for flat spacetime. The radiation propagates on the null cone, i.e., satisfies Huygens' principle.

It is emphasized that when we refer to a charged particle accelerated in a gravitational field, we are *not* referring to the equivalence principle. There is a well-known paradox of the equivalence principle which may be stated as follows. An insulated charged metal ball sitting on a table radiates no power because it is stationary, but it radiates power because the table accelerates it upward with acceleration g relative to an inertial frame in which the earth's gravity vanishes. This paradox has been discussed in great detail by Rohrlich (1965) and Boulware (1980) and will be ignored here. The application being considered here is where the acceleration of the charged particle is large compared with gravity. For example, an electron at a temperature of 10^6 K in a magnetic field of 10^6 T in the atmosphere of a neutron star of mass $1.4M_\odot$ and radius 15 km experiences a magnetic acceleration of order 10^{25} m/sec^2, many times the gravitational acceleration of order 10^{12} m/sec^2.

It is appropriate to comment on results in the literature concerning electromagnetic radiation from a charged particle in curved spacetime. Riesz (1948) in Chapter VI derived the Lienard–Wiechert potential in flat spacetime. Couch and Halliday (1971) considered radiation from a charged particle, but their gravitational field is due to the particle rather than being an ambient field as here. Günther (1965) concluded that if Maxwell's equations satisfy Huygens' principle, then the Bach tensor (Günther, 1988, p. 581), which depends on the tensor $L_{\mu\nu} = \frac{1}{6}Rg_{\mu\nu} - R_{\mu\nu}$, must vanish. DeWitt and Brehme (1960) obtained an expression [their equation (3.50)] for the electromagnetic vector potential of a charged particle in curved spacetime and differentiated it to obtain an expression [their equation (3.52)] for the electromagnetic field tensor of a charged particle, but the specific application made in the present paper was not pursued there. It may be noted from their equation (2.59) that a necessary condition for the tail term to vanish, i.e., for the electromagnetic vector potential to satisfy Huygens' principle, is that the tensor $L_{\mu\nu}$ vanish (therefore spacetime be empty). This tensor plays a key role in many of the necessary conditions for Huygens' principle to be satisfied [Carminati and McLenaghan (1986), equations (1.6)–(1.8)].

The principal application of this paper is to electromagnetic radiation from compact sources such as pulsar radiation and X-rays from black-hole accretion. The radiation is subject to the usual effects—Einstein deflection, Shapiro delay, and gravitational lensing. But it is null radiation satisfying Huygens' principle. A sharp pulse of radiation emitted at the source will remain sharp, rather than suffering a dispersion in velocity. These results are valid to first order in the Riemann tensor. An avenue for further inquiry would be to carry the calculations to second order in the Riemann tensor.

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